In[25]:= **Remove["Global`*"]**

Quadrotor Trajectory Generation and Control by M. Hehn and R. D'Andrea

Section 4.4: Optimal solutions

The analytical integration of the equations of motion is easy to perform by using piecewise integration between the switch times:

In[26]:=
$$x_{dd,1} = x_{dd,0} + t_1 * u$$
$$x_{dd,2} = x_{dd,1}$$
$$x_{dd,3} = x_{dd,2} - (t_3 - t_2) * u$$
$$x_{dd,4} = x_{dd,3}$$
$$x_{dd,f} = x_{dd,4} + (t_f - t_4) * u$$

Out[26]= $u\, t_1 + x_{dd,0}$

Out[27]= $u\, t_1 + x_{dd,0}$

Out[28]= $u\, t_1 - u\,(-t_2 + t_3) + x_{dd,0}$

Out[29]= $u\, t_1 - u\,(-t_2 + t_3) + x_{dd,0}$

Out[30]= $u\, t_1 - u\,(-t_2 + t_3) + u\,(-t_4 + t_f) + x_{dd,0}$

In[31]:=
$$x_{d,1} = x_{d,0} + x_{dd,0} * t_1 + 1/2 * u * t_1\text{^}2$$
$$x_{d,2} = x_{d,1} + x_{dd,1} * (t_2 - t_1)$$
$$x_{d,3} = x_{d,2} + x_{dd,2} * (t_3 - t_2) - 1/2 * u * (t_3 - t_2)\text{^}2$$
$$x_{d,4} = x_{d,3} + x_{dd,3} * (t_4 - t_3)$$
$$x_{d,f} = x_{d,4} + x_{dd,4} * (t_f - t_4) + 1/2 * u * (t_f - t_4)\text{^}2$$

Out[31]= $$\frac{u\, t_1^2}{2} + x_{d,0} + t_1\, x_{dd,0}$$

Out[32]= $$\frac{u\, t_1^2}{2} + x_{d,0} + t_1\, x_{dd,0} + (-t_1 + t_2)\,(u\, t_1 + x_{dd,0})$$

Out[33]= $$\frac{u\, t_1^2}{2} - \frac{1}{2} u\,(-t_2 + t_3)^2 + x_{d,0} + t_1\, x_{dd,0} + (-t_1 + t_2)\,(u\, t_1 + x_{dd,0}) + (-t_2 + t_3)\,(u\, t_1 + x_{dd,0})$$

Out[34]= $$\frac{u\, t_1^2}{2} - \frac{1}{2} u\,(-t_2 + t_3)^2 + x_{d,0} + t_1\, x_{dd,0} + (-t_1 + t_2)\,(u\, t_1 + x_{dd,0}) + (-t_2 + t_3)\,(u\, t_1 + x_{dd,0}) + (-t_3 + t_4)\,(u\, t_1 - u\,(-t_2 + t_3) + x_{dd,0})$$

Out[35]= $$\frac{u\, t_1^2}{2} - \frac{1}{2} u\,(-t_2 + t_3)^2 + \frac{1}{2} u\,(-t_4 + t_f)^2 + x_{d,0} + t_1\, x_{dd,0} + (-t_1 + t_2)\,(u\, t_1 + x_{dd,0}) + (-t_2 + t_3)\,(u\, t_1 + x_{dd,0}) + (-t_3 + t_4)\,(u\, t_1 - u\,(-t_2 + t_3) + x_{dd,0}) + (-t_4 + t_f)\,(u\, t_1 - u\,(-t_2 + t_3) + x_{dd,0})$$

In[36]:=
```
x1 = x0 + xd,0 * t1 + 1 / 2 * xdd,0 * t1^2 + 1 / 6 * u * t1^3
x2 = x1 + xd,1 * (t2 - t1) + 1 / 2 * xdd,1 * (t2 - t1)^2
x3 = x2 + xd,2 * (t3 - t2) + 1 / 2 * xdd,2 * (t3 - t2)^2 - 1 / 6 * u * (t3 - t2)^3
x4 = x3 + xd,3 * (t4 - t3) + 1 / 2 * xdd,3 * (t4 - t3)^2
xf = x4 + xd,4 * (tf - t4) + 1 / 2 * xdd,4 * (tf - t4)^2 + 1 / 6 * u * (tf - t4)^3
```

Out[36]= $$\frac{u\,t_1^3}{6} + x_0 + t_1\,x_{d,0} + \frac{1}{2}\,t_1^2\,x_{dd,0}$$

Out[37]= $$\frac{u\,t_1^3}{6} + x_0 + t_1\,x_{d,0} + \frac{1}{2}\,t_1^2\,x_{dd,0} + \frac{1}{2}\,(-t_1+t_2)^2\,(u\,t_1 + x_{dd,0}) + (-t_1+t_2)\left(\frac{u\,t_1^2}{2} + x_{d,0} + t_1\,x_{dd,0}\right)$$

Out[38]= $$\frac{u\,t_1^3}{6} - \frac{1}{6}\,u\,(-t_2+t_3)^3 + x_0 + t_1\,x_{d,0} + \frac{1}{2}\,t_1^2\,x_{dd,0} + \frac{1}{2}\,(-t_1+t_2)^2\,(u\,t_1 + x_{dd,0}) + \frac{1}{2}\,(-t_2+t_3)^2\,(u\,t_1 + x_{dd,0}) + (-t_1+t_2)\left(\frac{u\,t_1^2}{2} + x_{d,0} + t_1\,x_{dd,0}\right) + (-t_2+t_3)\left(\frac{u\,t_1^2}{2} + x_{d,0} + t_1\,x_{dd,0} + (-t_1+t_2)\,(u\,t_1 + x_{dd,0})\right)$$

Out[39]= $$\frac{u\,t_1^3}{6} - \frac{1}{6}\,u\,(-t_2+t_3)^3 + x_0 + t_1\,x_{d,0} + \frac{1}{2}\,t_1^2\,x_{dd,0} + \frac{1}{2}\,(-t_1+t_2)^2\,(u\,t_1 + x_{dd,0}) + \frac{1}{2}\,(-t_2+t_3)^2\,(u\,t_1 + x_{dd,0}) + \frac{1}{2}\,(-t_3+t_4)^2\,(u\,t_1 - u\,(-t_2+t_3) + x_{dd,0}) + (-t_1+t_2)\left(\frac{u\,t_1^2}{2} + x_{d,0} + t_1\,x_{dd,0}\right) + (-t_2+t_3)\left(\frac{u\,t_1^2}{2} + x_{d,0} + t_1\,x_{dd,0} + (-t_1+t_2)\,(u\,t_1 + x_{dd,0})\right) + (-t_3+t_4)\left(\frac{u\,t_1^2}{2} - \frac{1}{2}\,u\,(-t_2+t_3)^2 + x_{d,0} + t_1\,x_{dd,0} + (-t_1+t_2)\,(u\,t_1 + x_{dd,0}) + (-t_2+t_3)\,(u\,t_1 + x_{dd,0})\right)$$

Out[40]= $$\frac{u\,t_1^3}{6} - \frac{1}{6}\,u\,(-t_2+t_3)^3 + \frac{1}{6}\,u\,(-t_4+t_f)^3 + x_0 + t_1\,x_{d,0} + \frac{1}{2}\,t_1^2\,x_{dd,0} + \frac{1}{2}\,(-t_1+t_2)^2\,(u\,t_1 + x_{dd,0}) + \frac{1}{2}\,(-t_2+t_3)^2\,(u\,t_1 + x_{dd,0}) + \frac{1}{2}\,(-t_3+t_4)^2\,(u\,t_1 - u\,(-t_2+t_3) + x_{dd,0}) + \frac{1}{2}\,(-t_4+t_f)^2\,(u\,t_1 - u\,(-t_2+t_3) + x_{dd,0}) + (-t_1+t_2)\left(\frac{u\,t_1^2}{2} + x_{d,0} + t_1\,x_{dd,0}\right) + (-t_2+t_3)\left(\frac{u\,t_1^2}{2} + x_{d,0} + t_1\,x_{dd,0} + (-t_1+t_2)\,(u\,t_1 + x_{dd,0})\right) + (-t_3+t_4)\left(\frac{u\,t_1^2}{2} - \frac{1}{2}\,u\,(-t_2+t_3)^2 + x_{d,0} + t_1\,x_{dd,0} + (-t_1+t_2)\,(u\,t_1 + x_{dd,0}) + (-t_2+t_3)\,(u\,t_1 + x_{dd,0})\right) + (-t_4+t_f)\left(\frac{u\,t_1^2}{2} - \frac{1}{2}\,u\,(-t_2+t_3)^2 + x_{d,0} + t_1\,x_{dd,0} + (-t_1+t_2)\,(u\,t_1 + x_{dd,0}) + (-t_2+t_3)\,(u\,t_1 + x_{dd,0}) + (-t_3+t_4)\,(u\,t_1 - u\,(-t_2+t_3) + x_{dd,0})\right)$$

The boundary arc constraints are as follows:

If both boundary arcs vanish:

In[41]:=
```
nocon = {t2 == t1, t4 == t3}
```

Out[41]= $\{t_2 == t_1,\ t_4 == t_3\}$

If the second boundary arc vanishes:

In[42]:= **upcon = {t1 == (amax - xdd,0) / u, t4 == t3}**

Out[42]= $\left\{t_1 == \frac{a_{max} - x_{dd,0}}{u},\ t_4 == t_3\right\}$

If the first boundary arc vanishes:

In[43]:= **locon = {t2 == t1, t5 == t4 + (0 - amin) / u}**

Out[43]= $\left\{t_2 == t_1,\ t_5 == -\frac{a_{min}}{u} + t_4\right\}$

If both boundary arcs exist:

In[44]:= **twocon = {t1 == (amax - xdd,0) / u, t3 == t2 + (amax - amin) / u}**

Out[44]= $\left\{t_1 == \frac{a_{max} - x_{dd,0}}{u},\ t_3 == \frac{a_{max} - a_{min}}{u} + t_2\right\}$

This completes the equations defining the optimal solution.

Section 4.5: Feasibility of Rotational Rate Inputs

the accelerations are piecewise linear, so the vector f is defined for one piece to be

In[45]:= **f = {cx + ux * t, cy + uy * t, cz + uz * t + g}**

Out[45]= $\{c_x + t\,u_x,\ c_y + t\,u_y,\ g + c_z + t\,u_z\}$

from this we can calculate $\overline{f}$, and its time-derivative:

In[46]:= **f̄ = f / Sqrt[f.f]**

Out[46]=
$$\left\{\frac{c_x + t\,u_x}{\sqrt{(c_x + t\,u_x)^2 + (c_y + t\,u_y)^2 + (g + c_z + t\,u_z)^2}},\ \frac{c_y + t\,u_y}{\sqrt{(c_x + t\,u_x)^2 + (c_y + t\,u_y)^2 + (g + c_z + t\,u_z)^2}},\ \frac{g + c_z + t\,u_z}{\sqrt{(c_x + t\,u_x)^2 + (c_y + t\,u_y)^2 + (g + c_z + t\,u_z)^2}}\right\}$$

In[47]:= **ḟ̄ = D[f̄, t]**

Out[47]=
$$\left\{-\frac{(c_x + t\,u_x)\left(2\,u_x\,(c_x + t\,u_x) + 2\,u_y\,(c_y + t\,u_y) + 2\,u_z\,(g + c_z + t\,u_z)\right)}{2\left((c_x + t\,u_x)^2 + (c_y + t\,u_y)^2 + (g + c_z + t\,u_z)^2\right)^{3/2}} + \frac{u_x}{\sqrt{(c_x + t\,u_x)^2 + (c_y + t\,u_y)^2 + (g + c_z + t\,u_z)^2}},\right.$$
$$-\frac{(c_y + t\,u_y)\left(2\,u_x\,(c_x + t\,u_x) + 2\,u_y\,(c_y + t\,u_y) + 2\,u_z\,(g + c_z + t\,u_z)\right)}{2\left((c_x + t\,u_x)^2 + (c_y + t\,u_y)^2 + (g + c_z + t\,u_z)^2\right)^{3/2}} + \frac{u_y}{\sqrt{(c_x + t\,u_x)^2 + (c_y + t\,u_y)^2 + (g + c_z + t\,u_z)^2}},$$
$$\left.-\frac{(g + c_z + t\,u_z)\left(2\,u_x\,(c_x + t\,u_x) + 2\,u_y\,(c_y + t\,u_y) + 2\,u_z\,(g + c_z + t\,u_z)\right)}{2\left((c_x + t\,u_x)^2 + (c_y + t\,u_y)^2 + (g + c_z + t\,u_z)^2\right)^{3/2}} + \frac{u_z}{\sqrt{(c_x + t\,u_x)^2 + (c_y + t\,u_y)^2 + (g + c_z + t\,u_z)^2}}\right\}$$

Extrema of the rotational rate control inputs exist at the switching times, or at points where $\frac{d}{dt}\|\dot{\overline{f}}\|=0$ holds:

In[48]:=
```
FullSimplify[Solve[D[Sqrt[ḟ[[1]]^2 + ḟ[[2]]^2 + ḟ[[3]]^2], t] == 0, t]]
```

Out[48]= $\left\{\left\{t \to -\frac{c_x u_x + c_y u_y + (g + c_z) u_z}{u_x^2 + u_y^2 + u_z^2}\right\}\right\}$